Problem 71.1

Experiments in the Lincoln Tunnel (combined with the theoretical work discussed in exercise 63.7) suggest that the traffic flow is approximately

\[ q(\rho) = a\rho[\ln(\rho_{\text{max}}) - \ln(\rho)] \]

(where \(a\) and \(\rho_{\text{max}}\) are known constants). Suppose the initial density \(\rho(x,0)\) varies linearly from bumper-to-bumper traffic (behind \(x = -x_0\)) to no traffic (ahead of \(x = 0\)) as sketched in Fig. 71-6. Two hours later, where does \(\rho = \rho_{\text{max}}/2\)?

The location where \(\rho = \rho_{\text{max}}/2\) will follow the characteristic curve from \(x = -\frac{x_0}{2}\), where initially \(\rho(x,0) = \rho_{\text{max}}/2\). The corresponding characteristic curve has speed

\[ c = \frac{dq}{d\rho} = \frac{d}{d\rho}(a\rho[\ln(\rho_{\text{max}}) - \ln(\rho)]) = a(\ln(\rho_{\text{max}}) - \ln(\rho) - 1) = a(\ln(\rho_{\text{max}}) - \ln(\rho_{\text{max}}/2) - 1) = a(\ln(2) - 1). \]

After two hours \((t = 2)\), the density will be \(\rho = \rho_{\text{max}}/2\) at \(x = -\frac{x_0}{2} + at(\ln(2) - 1) = -\frac{x_0}{2} + 2a(\ln(2) - 1)\).

Problem 71.9

Show that \(\rho = f(x - q'(\rho)t)\) satisfies equation 71.1 for any function \(f\). Note that initially \(\rho = f(x)\). Briefly explain how this solution was obtained.
\[
\begin{align*}
\frac{\partial \rho}{\partial t} &= \frac{\partial}{\partial t} f(x - q'(\rho)t) \\
&= f'(x - q'(\rho)t) \left( -q'(\rho) - tq''(\rho) \frac{\partial \rho}{\partial t} \right) \\
(1 + tq''(\rho)) \frac{\partial \rho}{\partial t} &= -f'(x - q'(\rho)t)q'(\rho) \\
\frac{\partial \rho}{\partial t} &= -\frac{f'(x - q'(\rho)t)q'(\rho)}{1 + tq''(\rho)} \\
\frac{\partial \rho}{\partial x} &= \frac{\partial}{\partial x} f(x - q'(\rho)t) \\
&= f'(x - q'(\rho)t) \left( 1 - tq''(\rho) \frac{\partial \rho}{\partial x} \right) \\
(1 + tq''(\rho)) \frac{\partial \rho}{\partial x} &= f'(x - q'(\rho)t) \\
\frac{\partial \rho}{\partial x} &= f'(x - q'(\rho)t) \\n\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho} \frac{\partial \rho}{\partial x} &= \left( -\frac{f'(x - q'(\rho)t)q'(\rho)}{1 + tq''(\rho)} \right) + q'(\rho) \left( \frac{f'(x - q'(\rho)t)}{1 + tq''(\rho)} \right) \\
&= f'(x - q'(\rho)t) - (q'(\rho) + q'(\rho)) \\
&= 0
\end{align*}
\]

This solution was obtained by following characteristics forward in time. This formula is not as tidy as it first seems, however. Note that the definition of \( \rho \) contains \( \rho \) itself. This is an implicit definition of the solution \( \rho \), and obtaining \( \rho \) will in general require the solution to a nonlinear equation.

**Problem 72.5**

Sketch \( dq/d\rho \) as a function of \( x \), for fixed \( t > 0 \) (after the light turns green).

Let \( a = \left. \frac{dq}{d\rho} \right|_{\rho = \rho_{\text{max}}} = \rho_{\text{max}} u'(\rho_{\text{max}}) < 0 \). The characteristics corresponding to \( \rho = \rho_0 \) travel with velocity \( \frac{dx}{dt} = a \). Thus, for \( x < at \), \( \frac{dq}{d\rho} = a \).

Let \( b = \left. \frac{dq}{d\rho} \right|_{\rho = 0} = u_{\text{max}} > 0 \). The characteristics corresponding to \( \rho = 0 \) travel with velocity \( b \). For \( x > bt \), \( \frac{dq}{d\rho} = b \).

For \( at < x < bt \), the characteristics must emanate from \( x = 0 \) with densities between 0 and \( \rho_{\text{max}} \). Choose one such density, \( \rho_1 \). The characteristic corresponding to this density is \( \frac{dx}{dt} = \left. \frac{dq}{d\rho} \right|_{\rho = \rho_1} = q'(\rho_1) \), so that at \( x = q'(\rho_1)t \), the density will be \( \rho_1 \) and \( \frac{dq}{d\rho} = q'(\rho_1) = \frac{x}{t} \). That is, the profile for \( at < x < bt \) is the line \( \frac{dq}{d\rho} = \frac{x}{t} \). Note that the graph is three straight lines, even if \( q(\rho) \) is very complicated.
Problem 73.1

Assume that the traffic density is initially

\[
\rho(x,0) = \begin{cases} 
\rho_{\text{max}} & x < 0 \\
\rho_{\text{max}}/2 & 0 < x < a \\
0 & a < x.
\end{cases}
\]

Sketch the initial density. Determine and sketch the density at all later times. Assume that \( u = u_{\text{max}}(1 - \rho/\rho_{\text{max}}) \).

First, let's get the regions of constant density. The characteristics initially at \( x < 0 \) move with velocity \( \frac{dq}{d\rho} \bigg|_{\rho=\rho_{\text{max}}} = -u_{\text{max}}, \) so that for \( x < -u_{\text{max}}t \), \( \rho(x,t) = \rho_{\text{max}} \). The characteristics initially at \( 0 < x < a \) move with velocity \( \frac{dq}{d\rho} \bigg|_{\rho=\rho_{\text{max}}/2} = 0, \) so that for \( 0 < x < a \), \( \rho(x,t) = \frac{\rho_{\text{max}}}{2} \) for all time. The characteristics initially at \( x > a \) move with velocity \( \frac{dq}{d\rho} \bigg|_{\rho=0} = u_{\text{max}}, \) so that for \( x > a + u_{\text{max}}t \), \( \rho(x,t) = 0 \).

The first rarefaction fan corresponds to \( \rho_{\text{max}}/2 < \rho < \rho_{\text{max}} \) and initiates from \( x = 0 \). For such a \( \rho \), the characteristic moves with velocity \( \frac{dq}{d\rho} = u_{\text{max}} \left( 1 - \frac{2\rho}{\rho_{\text{max}}} \right) \). Thus, the density will be \( \rho \) at \( x = u_{\text{max}} \left( 1 - \frac{2\rho}{\rho_{\text{max}}} \right) t \). Solving for \( \rho \) yields \( \rho = \frac{\rho_{\text{max}}}{2} \left( 1 - \frac{x}{u_{\text{max}} t} \right) \), which is valid for \( -u_{\text{max}} t < x < 0 \).

The second rarefaction fan corresponds to \( 0 < \rho < \frac{\rho_{\text{max}}}{2} \) and initiates from \( x = a \). For such a \( \rho \), the characteristic moves with velocity \( \frac{dq}{d\rho} = u_{\text{max}} \left( 1 - \frac{2\rho}{\rho_{\text{max}}} \right) \). Thus, the density will be \( \rho \) at \( x = a + u_{\text{max}} \left( 1 - \frac{2\rho}{\rho_{\text{max}}} \right) t \). Solving for \( \rho \) yields \( \rho = \frac{\rho_{\text{max}}}{2} \left( 1 - \frac{x-a}{u_{\text{max}} t} \right) \), which is valid for \( a < x < a + u_{\text{max}} t \).
Problem 73.6

Assume

\[ u = u_{\text{max}} \left( 1 - \frac{\rho^2}{\rho_{\text{max}}^2} \right). \]

Determine the traffic density that results after an infinite line of stopped traffic is started by a red traffic light turning green.

From problem 72.5,

\[ \frac{dq}{d\rho} = \begin{cases} 
\rho_{\text{max}} u'(\rho_{\text{max}}) & x < \rho_{\text{max}} u'(\rho_{\text{max}}) t \\
\frac{\rho}{t} & \rho_{\text{max}} u'(\rho_{\text{max}}) t < x < u_{\text{max}} t \\
u_{\text{max}} & x > u_{\text{max}} t
\end{cases} \]

\[ u(\rho) = u_{\text{max}} \left( 1 - \frac{\rho^2}{\rho_{\text{max}}^2} \right) \]

\[ u'(\rho) = -\frac{2u_{\text{max}} \rho}{\rho_{\text{max}}^2} \]

\[ u'(\rho_{\text{max}}) = -\frac{2u_{\text{max}}^2}{\rho_{\text{max}}^3} = -\frac{2u_{\text{max}}}{\rho_{\text{max}}} \]

\[ \rho_{\text{max}} u'(\rho_{\text{max}}) = -2u_{\text{max}} \]

\[ q(\rho) = \rho u(\rho) \]

\[ q'(\rho) = u(\rho) + \rho u'(\rho) \]

\[ = u_{\text{max}} \left( 1 - \frac{\rho^2}{\rho_{\text{max}}^2} \right) - \frac{2u_{\text{max}} \rho^2}{\rho_{\text{max}}^3} \]

\[ = u_{\text{max}} \left( 1 - \frac{3 \rho^2}{\rho_{\text{max}}^2} \right) \]

With this, the above simplifies to

\[ \frac{dq}{d\rho} = \begin{cases} 
-2u_{\text{max}} & x < -2u_{\text{max}} t \\
\frac{\rho}{t} & -2u_{\text{max}} t < x < u_{\text{max}} t \\
u_{\text{max}} & x > u_{\text{max}} t
\end{cases} \]
Next, we need to compute $\rho$ from $q'(\rho)$.

$$1 - \frac{3\rho^2}{\rho_{\text{max}}^2} = \frac{q'(\rho)}{u_{\text{max}}}$$
$$\frac{3\rho^2}{\rho_{\text{max}}^2} = 1 - \frac{q'(\rho)}{u_{\text{max}}}$$
$$\frac{\rho^2}{\rho_{\text{max}}^2} = \frac{1}{3} \left(1 - \frac{q'(\rho)}{u_{\text{max}}} \right)$$
$$\frac{\rho}{\rho_{\text{max}}} = \sqrt{\frac{1}{3} \left(1 - \frac{q'(\rho)}{u_{\text{max}}} \right)}$$

$$\rho = \rho_{\text{max}} \sqrt{\frac{1}{3} \left(1 - \frac{q'(\rho)}{u_{\text{max}}} \right)}$$

$$\rho_1 = \rho_{\text{max}} \sqrt{\frac{1}{3} \left(1 - \frac{-2u_{\text{max}}}{u_{\text{max}}} \right)}$$
$$= \rho_{\text{max}}$$

$$\rho_2 = \rho_{\text{max}} \sqrt{\frac{1}{3} \left(1 - \frac{x}{u_{\text{max}}t} \right)}$$

$$\rho_3 = \rho_{\text{max}} \sqrt{\frac{1}{3} \left(1 - \frac{u_{\text{max}}}{u_{\text{max}}} \right)}$$
$$= 0$$

Now we can complete our solution

$$\rho = \begin{cases} 
\rho_{\text{max}} & x < -2u_{\text{max}}t \\
\rho_{\text{max}} \sqrt{\frac{1}{3} \left(1 - \frac{x}{u_{\text{max}}t} \right)} & -2u_{\text{max}}t < x < u_{\text{max}}t \\
0 & x > u_{\text{max}}t 
\end{cases}$$

**Problem 73.9**

At what velocity does the information that the traffic light changed from red to green travel?

Information about the traffic light reaches a location when the leading edge of the rarefaction fan arrives. This travels at velocity $u_{\text{max}}$. 
Problem 74.2

Assume \( u(\rho) = u_{\text{max}}(1 - \rho^2/\rho_{\text{max}}^2) \) and

\[
\rho(x, 0) = \begin{cases} 
\rho_{\text{max}} & x < 0 \\
\rho_{\text{max}}(L - x)/L & 0 < x < L \\
0 & L < x.
\end{cases}
\]

Determine \( \rho(x, t) \).

From problem 73.6, \( q'(0) = u_{\text{max}} \) and \( q'(\rho_{\text{max}}) = -2u_{\text{max}} \). Thus, the characteristics for \( \rho = \rho_{\text{max}} \) travel with velocity \(-2u_{\text{max}}\), and the characteristics for \( \rho = 0 \) travel with velocity \( u_{\text{max}} \). Thus, \( \rho = 0 \) for \( x > L + u_{\text{max}}t \) and \( \rho = \rho_{\text{max}} \) for \( x < -2u_{\text{max}} \).

Consider some other density \( \rho \), which is initially located \( (t = 0) \) at \( x(0) = L \left(1 - \frac{\rho}{\rho_{\text{max}}}\right) \). From problem 73.6, the characteristic speed for this density is \( c = q'(\rho) = u_{\text{max}} \left(1 - \frac{3\rho^2}{\rho_{\text{max}}^2}\right) \). Then, the density \( \rho \) at \( x = x(0) + ct = L \left(1 - \frac{\rho}{\rho_{\text{max}}}\right) + u_{\text{max}} \left(1 - \frac{3\rho^2}{\rho_{\text{max}}^2}\right)t \). This can be solved for \( \rho \).

\[
L \left(1 - \frac{\rho}{\rho_{\text{max}}}\right) + u_{\text{max}} \left(1 - \frac{3\rho^2}{\rho_{\text{max}}^2}\right)t = x
\]

\[
3u_{\text{max}}t \frac{\rho^2}{\rho_{\text{max}}^2} + L \frac{\rho}{\rho_{\text{max}}} + (x - L - u_{\text{max}}t) = 0
\]

\[
\frac{\rho}{\rho_{\text{max}}} = \frac{-L \pm \sqrt{L^2 - 4(3u_{\text{max}}t)(x - L - u_{\text{max}}t)}}{2(3u_{\text{max}}t)}
\]

\[
\rho = \frac{\rho_{\text{max}}}{6} \left(- \frac{L}{u_{\text{max}}t} + \sqrt{\left( \frac{L}{u_{\text{max}}t} \right)^2 - 12 \left( \frac{x - L}{u_{\text{max}}t} - 1 \right)} \right)
\]

The sign is chosen to keep the density positive. This density is valid for \(-2u_{\text{max}}t < x < L + u_{\text{max}}t\). Note that you can partially check your answer by plugging in the endpoints of this interval and verifying continuity with the rest of the solution.
Problem 74.3

Consider the following partial differential equation:

\[ \frac{\partial \rho}{\partial t} - \rho^2 \frac{\partial \rho}{\partial x} = 0 \quad -\infty < x < \infty. \]

(a) Why can’t this equation model a traffic flow problem?

\[ q'(\rho) = -\rho^2 \]
\[ q(\rho) = -\frac{1}{3}\rho^3 + c \]
\[ q(0) = c = 0 \]
\[ q(\rho) = -\frac{1}{3}\rho^3 \]
\[ u(\rho) = -\frac{1}{3}\rho^2 \]

The left-moving traffic (negative \( u \)) is not really an issue. What objectionable is that \( u(0) = 0 \) (cars driving in the absence of traffic don’t move) and the unboundedness of \( u \) (traffic moves faster as the road become more congested.)

(b) Solve this partial differential equation by the method of characteristics, subject to the initial conditions:

\[ \rho(x, 0) = \begin{cases} 
1 & x < 0 \\
1-x & 0 < x < 1 \\
0 & 1 < x.
\end{cases} \]

Consider the characteristics for \( \rho = 0 \). Here, \( q'(\rho) = 0 \), so the characteristics are constant. \( \rho = 0 \) whenever \( x > 1 \). For the case \( \rho = 1 \), the characteristics travel with velocity \( c = -1 \), so if \( x < -t \) the density will be \( \rho = 1 \). Otherwise, choose a density \( \rho \) in between. Then, \( c = -\rho^2 \) and \( x_0 = 1 - \rho \) so that at \( x = 1 - \rho - \rho^2 t \) the density will be \( \rho \). Solving for \( \rho \) gives \( \rho = \frac{-1 + \sqrt{1 - 4(x - 1)}}{2(1-x)} \), where the sign is chosen so \( \rho > 0 \). The behavior at \( t = 0 \) is more obvious if written as \( \rho = \frac{-1 + \sqrt{1 + 4(1-x)}}{2(1-x)} \).
Problem 74.4

Consider the example solved in this section. What traffic density should be approached as \( L \to 0 \)? Verify that as \( L \to 0 \) equation 74.9 approaches the correct traffic density.

The solution should match that derived in section 73.

\[
\rho = \begin{cases} 
\rho_{\text{max}} & x < -u_{\text{max}}t \\
\frac{\rho_{\text{max}}}{2} \left( 1 - \frac{x}{u_{\text{max}}t} \right) & -u_{\text{max}}t < x < u_{\text{max}}t \\
0 & x > u_{\text{max}}t 
\end{cases}
\]

Starting from equation 74.9,

\[
\rho(x, t) = \rho_{\text{max}} \left( 1 - \sqrt{1 - \frac{8u_{\text{max}}t(x - L - u_{\text{max}}t)}{L^2}} \right)^2
\]

\[
= \rho_{\text{max}} \left( 1 - \sqrt{1 - \frac{8u_{\text{max}}t(x - L - u_{\text{max}}t)}{L^2}} \right)^2
\]

\[
= \rho_{\text{max}} \left( L - L \sqrt{1 - \frac{8u_{\text{max}}t(x - L - u_{\text{max}}t)}{L^2}} \right)^2
\]

\[
= \rho_{\text{max}} \left( L - \sqrt{L^2 - 8u_{\text{max}}t(x - L - u_{\text{max}}t)} \right)^2
\]

\[
= \rho_{\text{max}} \frac{8u_{\text{max}}t(u_{\text{max}}t - x)}{16u_{\text{max}}^2t^2}
\]

\[
= \rho_{\text{max}} \frac{u_{\text{max}}t - x}{2u_{\text{max}}t}
\]

\[
= \frac{\rho_{\text{max}}}{2} \left( 1 - \frac{x}{u_{\text{max}}t} \right)
\]

This matches the fan portion, as expected.

Problem 77.1

If \( u = u_{\text{max}}(1 - \rho/\rho_{\text{max}}) \), then what is the velocity of a traffic shock separating densities \( \rho_0 \) and \( \rho_1 \)? (Simplify the expression as much as possible.) Show that the shock velocity is the average of the density wave velocities associated with \( \rho_0 \) and \( \rho_1 \).
The shock speed $s$ can be computed as

\[
q_1 = \rho_1 u(\rho_1) \\
q_2 = \rho_2 u(\rho_2) \\
s = \frac{q_1 - q_2}{\rho_1 - \rho_2} \\
= \frac{\rho_1 u_{\text{max}} \left(1 - \frac{\rho_1}{\rho_{\text{max}}}ight) - \rho_2 u_{\text{max}} \left(1 - \frac{\rho_2}{\rho_{\text{max}}}ight)}{\rho_1 - \rho_2} \\
= u_{\text{max}} - \frac{\rho_1 - \rho_2}{\rho_{\text{max}}} \left(\frac{\rho_1}{\rho_{\text{max}}} - \frac{\rho_2}{\rho_{\text{max}}}ight) \\
= u_{\text{max}} - \frac{\rho_1 + \rho_2}{\rho_{\text{max}}} \\
= \frac{\rho_1 + \rho_2}{\rho_{\text{max}}} \\
= s
\]

The density waves associated with $\rho_1$ and $\rho_2$ have velocities $c_1$ and $c_2$, where

\[
c = (\rho u(\rho))' \\
= \rho u'(\rho) + u(\rho) \\
= -\frac{u_{\text{max}}}{\rho_{\text{max}}} \left(1 - \frac{\rho}{\rho_{\text{max}}}ight) \\
= u_{\text{max}} \left(1 - \frac{2\rho}{\rho_{\text{max}}}ight) \\
= u_{\text{max}} \left(1 - \frac{2\rho_1}{\rho_{\text{max}}}ight) \\
= u_{\text{max}} \left(1 - \frac{2\rho_2}{\rho_{\text{max}}}ight) \\
= u_{\text{max}} \left(1 - \frac{\rho_1 + \rho_2}{\rho_{\text{max}}}ight) \\
= s
\]

**Problem 77.2**

If $u = u_{\text{max}}(1 - \rho^2/\rho_{\text{max}}^2)$, then what is the velocity of a traffic shock separating densities $\rho_0$ and $\rho_1$? (Simplify the expression as much as possible.) Show that the shock velocity is not the average of the density wave velocities associated with $\rho_0$ and $\rho_1$. 

The shock speed $s$ can be computed as

$$q_1 = \rho_1 u(\rho_1)$$
$$= \rho_1 u_{\text{max}} \left( 1 - \frac{\rho_1^2}{\rho_{\text{max}}^2} \right)$$

$$q_2 = \rho_2 u(\rho_2)$$
$$= \rho_2 u_{\text{max}} \left( 1 - \frac{\rho_2^2}{\rho_{\text{max}}^2} \right)$$

$$s = \frac{q_1 - q_2}{\rho_1 - \rho_2}$$
$$= \frac{\rho_1 u_{\text{max}} \left( 1 - \frac{\rho_1^2}{\rho_{\text{max}}^2} \right) - \rho_2 u_{\text{max}} \left( 1 - \frac{\rho_2^2}{\rho_{\text{max}}^2} \right)}{\rho_1 - \rho_2}$$

The density waves associated with $\rho_1$ and $\rho_2$ have velocities $c_1$ and $c_2$, where

$$c = (\rho u(\rho))^\prime$$
$$= \rho u'(\rho) + u(\rho)$$
$$= -\frac{2\rho u_{\text{max}}}{\rho_{\text{max}}^2} + u_{\text{max}} \left( 1 - \frac{\rho^2}{\rho_{\text{max}}^2} \right)$$

$$c_1 = u_{\text{max}} \left( 1 - \frac{3\rho_1^2}{\rho_{\text{max}}^2} \right)$$
$$c_2 = u_{\text{max}} \left( 1 - \frac{3\rho_2^2}{\rho_{\text{max}}^2} \right)$$

$$\frac{c_1 + c_2}{2} - s = \frac{1}{2} u_{\text{max}} \left( 1 - \frac{3\rho_1^2}{\rho_{\text{max}}^2} \right) + \frac{1}{2} u_{\text{max}} \left( 1 - \frac{3\rho_2^2}{\rho_{\text{max}}^2} \right) - u_{\text{max}} \left( 1 - \frac{\rho_1^2 + \rho_2^2}{\rho_{\text{max}}^2} \right)$$
$$= \frac{1}{2} u_{\text{max}} \left( -\frac{3\rho_1^2}{\rho_{\text{max}}^2} \right) - \frac{1}{2} u_{\text{max}} \left( -\frac{3\rho_2^2}{\rho_{\text{max}}^2} \right) - u_{\text{max}} \left( -\frac{\rho_1^2 + \rho_2^2}{\rho_{\text{max}}^2} \right)$$
$$= -\frac{u_{\text{max}}}{2\rho_{\text{max}}^2} (3\rho_1^2 - 3\rho_2^2 + 2(\rho_1^2 + \rho_2^2))$$
$$= -\frac{u_{\text{max}}}{2\rho_{\text{max}}^2} (\rho_1 - \rho_2)^2$$

This vanishes only if $\rho_1 = \rho_2$. 

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Problem 77.3

A weak shock is a shock in which the shock strength (the difference in densities) is small. For a weak shock, show that the shock velocity is approximately the average of the density wave velocities associated with the two densities. [Hint: Use Taylor series methods.]

Let $\rho_1 = \bar{\rho} - \epsilon$ and $\rho_2 = \bar{\rho} + \epsilon$, where $\epsilon \ll \bar{\rho}$. Let $\bar{u} = u(\bar{\rho})$, $\bar{u}' = u'(\bar{\rho})$, etc. First, let’s work out some useful Taylor series expansions.

\[
\begin{align*}
    u_2 &= u(\rho_2) \\
          &= u(\bar{\rho} + \epsilon) \\
          &= \bar{u} + \epsilon \bar{u}' + \frac{1}{2} \epsilon^2 \bar{u}'' + O(\epsilon^3) \\
    u_1 &= \bar{u} - \epsilon \bar{u}' + \frac{1}{2} \epsilon^2 \bar{u}'' + O(\epsilon^3) \\
\end{align*}
\]

\[
\begin{align*}
    \frac{u_1 + u_2}{2} &= \bar{u} + \frac{1}{2} \epsilon^2 \bar{u}'' + O(\epsilon^3) \\
                      &= \bar{u} + O(\epsilon^2) \\
    \frac{u_2 - u_1}{2\epsilon} &= \bar{u}' + O(\epsilon^2) \\
    \frac{u'_2 + u'_1}{2} &= \bar{u}' + O(\epsilon^2) \\
    \frac{u'_2 - u'_1}{2\epsilon} &= \bar{u}'' + O(\epsilon) \\
\end{align*}
\]

Then, the shock speed is

\[
\begin{align*}
    s &= \frac{\rho_2 u_2 - \rho_1 u_1}{\rho_2 - \rho_1} \\
     &= \frac{(\bar{\rho} + \epsilon)u_2 - (\bar{\rho} - \epsilon)u_1}{(\bar{\rho} + \epsilon) - (\bar{\rho} - \epsilon)} \\
     &= \frac{(\bar{\rho} + \epsilon)u_2 - (\bar{\rho} - \epsilon)u_1}{2\epsilon} \\
     &= \frac{\bar{\rho}u_2 - \bar{\rho}u_1 + \epsilon u_2 + \epsilon u_1}{2\epsilon} \\
     &= \bar{\rho} \frac{u_2 - u_1}{2\epsilon} + \frac{u_2 + u_1}{2} \\
     &= \bar{\rho} \bar{u}' + \bar{u} + O(\epsilon^2) \\
\end{align*}
\]
The average of the density wave velocities is

\[
c = q' \rho \\
= (\rho u(\rho))' \\
= \rho u' \rho + u(\rho)
\]

\[
c_1 + c_2 = \frac{(\rho_1 u'(\rho_1) + u(\rho_1)) + (\rho_2 u'(\rho_2) + u(\rho_2))}{2}
\]

\[
= \frac{\rho_1 u'(\rho_1) + \rho_2 u'(\rho_2)}{2} + \frac{u(\rho_1) + u(\rho_2)}{2}
\]

\[
= \frac{\rho_1 u_1' + \rho_2 u_2'}{2} + \frac{u_1 + u_2}{2}
\]

\[
= \frac{(\bar{\rho} - \epsilon)u_1' + (\bar{\rho} + \epsilon)u_2'}{2} + \frac{u_1 + u_2}{2}
\]

\[
= \bar{\rho} u' + \epsilon^2 \bar{\rho} u'' + \bar{u} + O(\epsilon^2)
\]

\[
= \bar{\rho} u' + \bar{u} + O(\epsilon^2)
\]

\[
= s + O(\epsilon^2)
\]

This approximation is actually quite good.

**Additional Problem**

A pulley of radius \( r_1 \) hangs from a spring (rest length \( \ell \), spring constant \( k \)) from the ceiling of a room as shown at right. The center of the pulley is at \((0, y(t))\), and the top of the spring is fixed at \((0, \ell)\). The pulley is able to rotate around the end of the spring without friction. A cable extends around the pulley with a weight \( m_1 \) at one end (located at \((-r_1, z(t))\)) and the other end fixed to the floor at \((r_1, -b)\). Two identical point masses \( m_2 \) are attached to the pulley at a distance \( r_2 \) from the pulley’s center at angles \( \theta(t) \) and \( \theta(t) + \pi \). Assume the cable has negligible mass. When the spring is at its rest length, \( y(t) = 0 \), \( z(t) = -a \), and \( \theta(t) = \frac{\pi}{4} \). You may assume that \( m_1 \) never touches the pulley or the ground.

(a) Express \( z(t) \) and \( \theta(t) \) in terms of \( y(t) \).

The rope consists of three pieces: the part attached to the floor (length \( y(t) + b \)), the portion contacting the pulley (length \( \pi r_1 \)), and the part attached to the weight (length \( y(t) - z(t) \)). The total length is \( y(t) + b + \pi r_1 + y(t) - z(t) = a + b + \pi r_1 \), so that \( z(t) = 2y(t) - a \).

To get \( \theta(t) \), assume the part of the pulley at \( \theta(t) \) remains in contact with the rope. The length of rope between the ground and this point on the pulley is \( \ell_2 = y(t) + b + \theta(t)r_1 \), and this does not change as long as this point on the pulley does not separate from the rope (or slip). We are given that when \( y(t) = 0 \) we have \( \theta(t) = \frac{\pi}{4} \), so that \( \ell_2 = b + \frac{\pi r_1}{4} \). Then, \( b + \frac{\pi r_1}{4} = \ell_2 = y(t) + b + \theta(t)r_1 \) leads to \( \theta(t) = \frac{\pi}{4} - \frac{y(t)}{r_1} \).

(b) Express the potential energy \( \phi_1(t) \) and kinetic energy \( K_1(t) \) of the mass \( m_1 \) in terms of \( y(t) \) and \( y'(t) \).
The kinetic energy is $K_1(t) = \frac{1}{2}m_1z'(t)^2 = 2m_1y'(t)^2$. The potential energy is $\phi_1(t) = m_1g(z(t) = m_1g(2y(t) - a)$.

(c) Express the potential energy $\phi_2(t)$ and kinetic energy $K_2(t)$ of the two masses $m_2$ in terms of $y(t)$ and $y'(t)$.

Both masses move with speed $\theta'(t)r_2$, so their kinetic energy will be

$$K_2(t) = \frac{1}{2}m_2(\theta'(t)r_2)^2 + \frac{1}{2}m_2(\theta'(t)r_2)^2 = \frac{m_2r_2^2}{r_1^2}y'(t)^2$$

The heights of the two masses are $y(t) + r_2\sin\theta(t)$ and $y(t) - r_2\sin\theta(t)$, so that the potential energy is

$$\phi_2(t) = m_2g(y(t) + r_2\sin\theta(t)) + m_2g(y(t) - r_2\sin\theta(t)) = 2m_2gy(t).$$

(d) Express the potential energy $\phi_3(t)$ of the spring in terms of $y(t)$.

The spring is displaced from its rest length by $y(t)$, so the potential energy is $\phi_3(t) = \frac{1}{2}ky(t)^2$.

(e) The total energy is $E = \phi_1 + \phi_2 + \phi_3 + K_1 + K_2$. Locate the equilibrium $y_{eq}$.

The equilibria occur when $\frac{\partial \phi}{\partial y} = 0$.

$$\phi = m_1g(2y - a) + 2m_2gy + \frac{1}{2}ky^2$$

$$0 = \frac{\partial \phi}{\partial y}$$

$$= 2(m_1 + m_2)g + ky_{eq}$$

$$y_{eq} = -\frac{2(m_1 + m_2)g}{k}$$

(f) Let $w(t) = y(t) - y_{eq}$, where $y_{eq}$ is the equilibrium configuration. Show that $w(t)$ is described by the same ODE $m\dddot{w} + \ddot{w} + kw = 0$ that was used to describe a simple spring. What are the constants $\dot{m}$, $\dot{c}$, and $\dot{k}$?
Beginning with total energy,

\[ E = m_1 g (2y(t) - a) + 2m_2 g y(t) + \frac{1}{2} ky(t)^2 + 2m_1 y'(t)^2 + \frac{m_2 r^2}{r_1^2} y'(t)^2 \]

\[ = \frac{1}{2} ky(t)^2 + 2(m_1 + m_2) g y(t) - m_1 g a + \left( 2m_1 + \frac{m_2 r^2}{r_1^2} \right) \frac{y'(t)^2}{y(t)^2} \]

\[ 0 = E' \]

\[ = ky(t) y'(t) + 2(m_1 + m_2) g y'(t) + 2 \left( 2m_1 + \frac{m_2 r^2}{r_1^2} \right) y''(t) y'(t) \]

\[ 0 = 2 \left( 2m_1 + \frac{m_2 r^2}{r_1^2} \right) y''(t) + ky(t) + 2(m_1 + m_2) g \]

\[ = 2 \left( 2m_1 + \frac{m_2 r^2}{r_1^2} \right) (w(t) + y_{eq})'' + k (w(t) + y_{eq}) + 2(m_1 + m_2) g \]

\[ = 2 \left( 2m_1 + \frac{m_2 r^2}{r_1^2} \right) w''(t) + kw(t) + ky_{eq} + 2(m_1 + m_2) g \]

\[ = 2 \left( 2m_1 + \frac{m_2 r^2}{r_1^2} \right) w''(t) + kw(t) \]

\[ \dot{m} = 4m_1 + \frac{2m_2 r^2}{r_1^2} \]

\[ \dot{c} = 0 \]

\[ \dot{k} = k \]