Math 142-2, Homework 4

Solutions

Problem 57.2

57.2. Suppose a velocity field is given:

\[ u(x, t) = \frac{30x + 30L}{15t + L} \]

(a) Determine the motion of a car which starts at \( x = L/2 \) at \( t = 0 \). [Hint: Why does \( \frac{dx}{dt} = \frac{30x + 30L}{15t + L} \)? Solve this differential equation. It is separable.]

Let \( x(t) \) be the position of a particular car at time \( t \). Since it is at position \( x(t) \) at time \( t \), its velocity at that time is \( u(x(t), t) \). Then, \( \frac{dx}{dt} = u(x(t), t) = \frac{30x + 30L}{15t + L} \).

\[
\int \frac{dx}{30x + 30L} = \int \frac{dt}{15t + L} \\
\frac{1}{30} \int \frac{dx}{x + L} = \frac{1}{15} \int \frac{dt}{t + \frac{L}{15}} \\
\frac{1}{30} \ln |x + L| = \frac{1}{15} \ln \left| t + \frac{L}{15} \right| + c_1 \\
\ln |x + L| = 2 \ln \left| t + \frac{L}{15} \right| + 30c_1 \\
|x + L| = e^{30c_1} \left( t + \frac{L}{15} \right)^2 \\
x + L = c_2 \left( t + \frac{L}{15} \right)^2 \\
x = c_2 \left( t + \frac{L}{15} \right)^2 - L \\
\frac{L}{2} = c_2 \left( 0 + \frac{L}{15} \right)^2 - L \\
\frac{1}{2} = c_2 \frac{L}{225} - 1 \\
c_2 = \frac{675}{2L}
\]
\[ x = c_2 \left( t + \frac{L}{15} \right)^2 - L = \frac{3}{2L} (15t + L)^2 - L \]

**(b)** Show that \( u(x, t) \) is constant along straight lines in the \( x - t \) plane, but the car does not move at a constant velocity.

Let \( c \) be some constant and \( u(x, t) = c \). Then,

\[
\begin{align*}
  c &= \frac{dx}{dt} \\
  &= \frac{30x + 30L}{15t + L} \\
  30x + 30L &= c(15t + L) \\
  x + L &= \frac{c}{30} (15t + L) \\
  x &= \frac{c}{30} (15t + L) - L
\end{align*}
\]

This is an equation for a straight line. The car does not move at constant velocity because its velocity is 

\[
v = \frac{dx}{dt} = \frac{d}{dt} \left( c_2 \left( t + \frac{L}{15} \right)^2 - L \right) = 2c_2 \left( t + \frac{L}{15} \right),
\]

which depends on time.

**Problem 57.3**

Suppose that the velocity field \( u(x, t) \) is known. What mathematical problem needs to be solved in order to determine the position of a car at later times, which starts (at \( t = 0 \)) at \( x = L \)?

The ODE \( \frac{dx}{dt} = u(x, t) \), subject to the initial condition \( x(0) = L \), must be solved for \( x(t) \).

**Problem 57.5**

Suppose that \( u(x, t) = e^{-t} \).

**(a)** Sketch curves in \( x - t \) space along which \( u(x, t) \) is constant.
(b) Determine the time dependence of the position of any car.

\[
\frac{dx}{dt} = e^{-t}
\]

\[
x = \int e^{-t} \, dt
\]

\[
x = -e^{-t} + c
\]

\[
x_0 = -1 + c
\]

\[
c = x_0 + 1
\]

\[
x = 1 - e^{-t} + x_0
\]

(c) In the same $x - t$ space used in part (a), sketch various different car paths.

Some paths are shown in blue.
Problem 57.6

Consider an infinite number of cars, each designated by a number $\beta$. Assume the car labeled $\beta$ starts from $x = \beta \ (\beta > 0)$ with zero velocity, and also assume it has a constant acceleration $\beta$.

(a) Determine the position and velocity of each car as a function of time.

\[
\frac{dv}{dt} = \beta \\
v(t) = \beta t + c \\
v(0) = c = 0 \\
v(t) = \beta t \\
\frac{dx}{dt} = \beta t \\
x(t) = \frac{1}{2} \beta t^2 + k \\
x(0) = k = \beta \\
x(t) = \frac{1}{2} \beta t^2 + \beta
\]

(b) Sketch the path of a typical car.

Several paths are shown in blue.

(c) Determine the velocity field $u(x, t)$.
We begin by fixing a \((x, t)\). First, we need to figure out which car \(\beta\) this represents.

\[
x = \frac{1}{2} \beta t^2 + \beta
\]
\[
\beta = \frac{2x}{t^2 + 2}
\]

For this car, the velocity is

\[
v = \beta t
\]
\[
v = \frac{2xt}{t^2 + 2}
\]
\[
u(x, t) = \frac{2xt}{t^2 + 2}
\]

(d) Sketch curves along which \(u(x, t)\) is a constant.

\[
c = u(x, t) = \frac{2xt}{t^2 + 2}
\]
\[
x = \frac{(t^2 + 2)c}{2t}
\]

Some curves of constant \(u(x, t)\) are shown in red.

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**Problem 59.2**

Suppose that at position \(x_0\) the traffic flow is known, \(q(x_0, t)\), and varies with time. Calculate the number of cars that pass \(x_0\), between \(t = 0\) and \(t = t_0\).

The traffic flow \(q(x_0, t)\) is the number of cars passing a fixed point \(x_0\) per unit time. That is, for some small time interval \(\Delta t\), the number of cars that pass \(x_0\) will be approximately \(\Delta t|q(x_0, t)|\), taking special
care of the fact that $q$ may be positive or negative (flow to the right or left respectively). The total number of cars will be obtained by adding these, which leads to the integral $N = \int_0^t |q(x_0, t)| \, dt$. Note the similarity to equation 60.1. If $q > 0$ is assumed, the absolute values can be omitted.

**Problem 59.3**

In an experiment the total number of cars that pass a position $x_0$ after $t = 0$, $M(x_0, t)$, is measured as a function of time. Assume this series of points has been smoothed to make a continuous curve.

(a) Briefly explain why the curve $M(x_0, t)$ is increasing as $t$ increases.

If $M(x_0, t)$ is taken to be the discrete, discontinuous function, then it simply represents the (integer) number of cars that have passed. The number of cars that pass a given point certainly cannot decrease (once a car has passed you, this event cannot be undone). Assuming cars are present, they will pass you from time to time, thus leading to $M(x_0, t)$ increasing.

The problem asks us to assume that $M(x_0, t)$ has been smoothed out to make it continuous. It is possible that in doing so, wiggles are introduced that make it decrease in places. Certainly there need not be any wiggles. One way to construct such a function $M(x_0, t)$ would be to record the times $t_1, t_2, \ldots, t_n$ when cars $1, 2, \ldots, n$ pass location $x_0$. Then, connect the points $(0, 0), (t_1, 1), (t_2, 2), \ldots, (t_n, n)$ with straight lines. This function is strictly increasing, since $0 < t_1 < t_2 < \cdots < t_n$. It is not unreasonable to assume then that $M(x_0, t)$ was smoothed in a way that does not cause it to decrease.

(b) What is the traffic flow at $t = \tau$?

Pick some small interval of time $[\tau, \tau + \Delta t]$. The number of cars that have crossed during this time is $\Delta N = M(x_0, \tau + \Delta t) - M(x_0, \tau)$. The traffic flow is
\[
q(x_0, \tau) = \lim_{\Delta t \to 0} \frac{\Delta N}{\Delta t} = \lim_{\Delta t \to 0} \frac{M(x_0, \tau + \Delta t) - M(x_0, \tau)}{\Delta t} = \frac{\partial}{\partial \tau} M(x_0, \tau).
\]

**Problem 60.1**

Consider a semi-infinite highway $0 \leq x < \infty$ (with no entrances or exits other than at $x = 0$). Show that the number of cars on the highway at time $t$ is
\[
N(t) = N_0 + \int_0^t q(0, \tau) \, d\tau,
\]
where $N_0$ is the number of cars on the highway at $t = 0$. (You may assume that $\rho(x, t) \to 0$ as $x \to \infty$.)

Consider a small interval of time $[\tau, \tau + \Delta t]$. If $q(0, t) > 0$ for $\tau < t < \tau + \Delta t$, then cars are moving to the right, increasing the number of cars on the semi-infinite highway. That is, approximately $\Delta N = \Delta q(0, \tau)$ cars are added to the highway. If $q(0, t) < 0$ for $\tau < t < \tau + \Delta t$, cars are moving to the left, leaving the highway and reducing the number of cars on it. In this case, $\Delta N = \Delta q(0, \tau) < 0$, indicating that $|\Delta N|$ cars have left. Either way, the change in the number of cars is $\Delta N$. Adding these increments up for the full length of time suggests that the final number of cars should be
\[
N(t) = N_0 + \int_0^t q(0, \tau) \, d\tau.
\]
The problem statement lets us assume that $\rho(x, t) \to 0$ as $x \to \infty$. If we also assume in addition that, for example the velocity $u(x, \tau)$ is bounded, then $q(x, \tau) = u(x, \tau) \rho(x, \tau) \to 0$ as $x \to \infty$. 

\[
0 = \int_0^t \int_0^\infty \frac{\partial}{\partial \tau} \rho(x, \tau) + \frac{\partial}{\partial x} q(x, \tau) \, dx \, d\tau
= \int_0^t \int_0^\infty \frac{\partial}{\partial \tau} \rho(x, \tau) \, dx \, d\tau + \int_0^t \int_0^\infty \frac{\partial}{\partial x} q(x, \tau) \, dx \, d\tau
= \int_0^\infty \int_0^t \frac{\partial}{\partial \tau} \rho(x, \tau) \, dx \, d\tau + \int_0^\infty \int_0^t \frac{\partial}{\partial x} q(x, \tau) \, dx \, d\tau
= \int_0^t \left( \rho(x, t) - \rho(x, 0) \right) \, dx + \int_0^t \left( \lim_{x \to \infty} q(x, \tau) - q(0, \tau) \right) \, d\tau
= \int_0^t \rho(x, t) \, dx - \int_0^\infty \rho(x, 0) \, dx - \int_0^t q(0, \tau) \, d\tau
= N(t) - N_0 - \int_0^t q(0, \tau) \, d\tau
\]

\[
N(t) = N_0 + \int_0^t q(0, \tau) \, d\tau
\]

**Problem 60.2**

Suppose that we are interested in the change in the number of cars $N(t)$ between two observers, one fixed at $x = a$ and the other moving in some prescribed manner, $x = b(t)$:

\[
N(t) = \int_a^{b(t)} \rho(x, t) \, dx
\]

(a) The derivative of an integral with a variable limit is

\[
\frac{dN}{dt} = \frac{db}{dt} \rho(b, t) + \int_a^{b(t)} \frac{\partial \rho}{\partial t} \, dx.
\]

(Note that the integrand, $\rho(x, t)$, also depends on $t$.) Show this result either by considering $\lim_{\Delta t \to 0} [N(t + \Delta t) - N(t)]/\Delta t$ or by using the chain rule for derivatives.
If instead $b(t) > b(t + \Delta t)$, the inequalities are reversed but the squeeze is the same. Since $\epsilon$ is arbitrarily small,

$$
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{b(t)}^{b(t + \Delta t)} \rho(x, t + \Delta t) \, dx = \lim_{\Delta t \to 0} \rho(b(t), t + \Delta t) \frac{1}{\Delta t} \int_{b(t)}^{b(t + \Delta t)} \rho(x, t) \, dx
$$

$$
= \rho(b(t), t) \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{b(t)}^{b(t + \Delta t)} \rho(x, t) \, dx
$$

$$
= \rho(b(t), t) \frac{b(t + \Delta t) - b(t)}{\Delta t}
$$

Finally,

$$
\frac{dN}{dt} = \int_a^b \frac{\partial \rho}{\partial t} \, dx + \rho(b, t) \frac{db}{dt}
$$
(b) Using $\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u)$ show that

$$\frac{dN}{dt} = -\rho(b, t) \left( u(b, t) - \frac{db}{dt} \right) + \rho(a, t) u(a, t).$$

$$\frac{dN}{dt} = \int_a^{b(t)} \frac{\partial \rho}{\partial t} \, dx + \rho(b, t) \frac{db}{dt}$$

$$= \int_a^{b(t)} - \frac{\partial}{\partial x}(\rho u) \, dx + \rho(b, t) \frac{db}{dt}$$

$$= \rho(a, t) u(a, t) - \rho(b, t) u(b, t) + \rho(b, t) \frac{db}{dt}$$

$$= -\rho(b, t) \left( u(b, t) - \frac{db}{dt} \right) + \rho(a, t) u(a, t)$$

(c) Interpret the result of part (b) if the moving observer is in a car moving with the traffic.

In this case, $\frac{db}{dt} = u(a, t)$, so

$$\frac{dN}{dt} = \rho(a, t) u(a, t) = q(a, t).$$

That is, the rate of change in the number of cars between $a$ and $b$ is the same as the rate at which cars pass the observer at $a$. No cars pass $b$. (Compare this with equation 60.2.)

**Problem 61.3**

Assume that a velocity field, $u(x, t)$, exists. Show that the acceleration of an individual car is given by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}.$$ 

Let $x(t)$ be the position of some car. Then,

$$v(t) = \frac{dx}{dt}(t) = u(x(t), t)$$

is its velocity. Finally, its acceleration is

$$a(t) = \frac{dv}{dt}(t) = \frac{\partial u}{\partial t}(x(t), t) + \frac{\partial u}{\partial x}(x(t), t) \frac{dx}{dt}(t) = \frac{\partial u}{\partial t}(x(t), t) + \frac{\partial u}{\partial x}(x(t), t) u(x(t), t).$$